Algebraic geometry 1

Exercise Sheet 12

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Exercise 1. Let $X \subset \mathbb{P}^2$ be the union of three points not lying on a line. Prove that the homogeneous ideal $I(X)$ of X can not be generated by two polynomials.
Hint: Use Bézout's Theorem.

Exercise 2. Let $X \subset \mathbb{P}^n$ be a projective variety of dimension ≥ 1, and let $F \in K[X_0, \ldots, X_n]$ be a non-constant homogeneous polynomial that does not vanish identically on X (that is: $V^p(F)$ does not contain X).
Show that every irreducible component of $X \cap V(F)$ has dimension $\dim X - 1$.
Hint: Deduce from Exercise 5, Sheet 10.

Exercise 3. (1) Let $X \subset \mathbb{P}^n$ be a projective curve (that is of pure dimension 1) of degree 1. Show that X is a line in $\mathbb{P}^n$.
Hint: Using Bézout's Theorem show the following observation: if a hyperplane $H \subset \mathbb{P}^n$ contains two points from X, then $X \subset H$.
(2) Generalize (1): Let $X \subset \mathbb{P}^n$ be a pure-dimensional projective algebraic set. Show that $\deg X = 1$ if and only if X is a linear subspace (variety) in $\mathbb{P}^n$.
Hint: From the lecture we know already that the degree of a linear subspace is 1. So it remains to show that if $\deg X = 1$ then X is a linear subspace. Show this by induction on $\dim X$. Exercise 2 may be useful for the induction step.

Exercise 4. (**Pappus's theorem**) Let l_1 and l_2 be two line in $\mathbb{P}^2$. Let A, B, C be three different points on l_1 and D, E, F be three different points on l_2.
Show that the three intersection points $Q = AE \cap BD$, $P = AF \cap CD$ and $R = BF \cap CE$ lie on a line.